

On q -Deformations of Clifford Algebras*

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Abstract

Several Clifford algebras that are covariant under the action of a Lie algebra \mathfrak{g} can be deformed in a way consistent with the deformation of $U\mathfrak{g}$ into a quantum group (or into a triangular Hopf algebra) $U_q\mathfrak{g}$, i.e. so as to remain covariant under the action of $U_q\mathfrak{g}$. In this report, after recalling these facts, we review our results regarding the formal realization of the elements of such “ q -deformed” Clifford algebras as “functions” (polynomials) in the generators of the undeformed ones; in particular, the intriguing interplay between the original and the q -deformed symmetry. Finally, we briefly illustrate their dramatic consequences on the representation theories of the original and of the q -deformed Clifford algebra, and mention how these results could turn out to be useful in quantum physics.

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1 Introduction

We first introduce the notions of a q -deformed Clifford algebra and of a deforming map in an explicit way on a simple example.

Consider the Clifford algebra \mathcal{A} generated by $\mathbf{1}, a^\uparrow, a^\downarrow, a_\uparrow^+, a_\downarrow^+$ fulfilling the anticommutation relations

$$\begin{aligned} a^i a^j + a^j a^i &= 0 \\ a_i^+ a_j^+ + a_j^+ a_i^+ &= 0 \quad i, j = \uparrow, \downarrow \\ a^i a_j^+ + a_j^+ a^i &= \delta_j^i \mathbf{1} \end{aligned} \quad (1.1)$$

When equipped with the $*$ -structure

$$(a^i)^* = a_i^+ \quad (1.2)$$

this becomes the familiar algebra of creation and annihilation operators of a fermionic system with two modes (e.g. a spin-up and a spin-down one-particle state). \mathcal{A} is then covariant under the action \triangleright of $su(2)$, i.e. is a $U su(2)$ -module algebra. This means that (1.1) are left invariant by the action of $su(2)$, which is given by the standard defining action of $su(2)$ on the generators a^i, a_i^+ and is extended to the whole \mathcal{A} according to linearity and the Leibniz rule

$$x \triangleright (\alpha \alpha') = (x \triangleright \alpha) \alpha' + \alpha (x \triangleright \alpha'). \quad (1.3)$$

We shall not assign the $*$ -structure for the moment, so \mathcal{A} will be covariant under the action of $sl(2, \mathbb{C})$.

\mathcal{A} is the simplest algebra one can q -deform: The corresponding q -deformed algebra \mathcal{A}_q ($q \in \mathbb{C} - \{0\}$) is generated by $\mathbf{1}_q, \tilde{A}^\uparrow, \tilde{A}^\downarrow, \tilde{A}_\uparrow^+, \tilde{A}_\downarrow^+$ fulfilling the quadratic anticommutation relations

$$\begin{aligned} \tilde{A}^\uparrow \tilde{A}^\uparrow &= 0 = \tilde{A}^\downarrow \tilde{A}^\downarrow \\ \tilde{A}_\uparrow^+ \tilde{A}_\uparrow^+ &= 0 = \tilde{A}_\downarrow^+ \tilde{A}_\downarrow^+ \\ \tilde{A}^\uparrow \tilde{A}^\downarrow + q \tilde{A}^\downarrow \tilde{A}^\uparrow &= 0 \\ \tilde{A}_\uparrow^+ \tilde{A}_\downarrow^+ + q^{-1} \tilde{A}_\downarrow^+ \tilde{A}_\uparrow^+ &= 0 \\ \tilde{A}^\uparrow \tilde{A}_\downarrow^+ + q^{-1} \tilde{A}_\downarrow^+ \tilde{A}^\uparrow &= 0 \\ \tilde{A}^\downarrow \tilde{A}_\uparrow^+ + q^{-1} \tilde{A}_\uparrow^+ \tilde{A}^\downarrow &= 0 \\ \tilde{A}^\uparrow \tilde{A}_\uparrow^+ + \tilde{A}_\uparrow^+ \tilde{A}^\uparrow &= \mathbf{1}_q + (q^{-1} - 1) \tilde{A}_\downarrow^+ \tilde{A}^\downarrow \\ \tilde{A}^\downarrow \tilde{A}_\downarrow^+ + \tilde{A}_\downarrow^+ \tilde{A}^\downarrow &= \mathbf{1}_q. \end{aligned} \quad (1.4)$$

This algebra was first introduced in ref. [14]. Clearly $\mathcal{A}_q \xrightarrow{q \rightarrow 1} \mathcal{A}$ if we identify $\tilde{A}^i \rightarrow a^i$, $\tilde{A}_i^+ \rightarrow a_i^+$ and $\mathbf{1}_q \rightarrow \mathbf{1}$ in the limit. Moreover, (1.4), and hence \mathcal{A}_q , are covariant under the action $\tilde{\triangleright}_q$ of the quantum group $U_q sl(2)$.

One can easily show that $\mathcal{A}, \mathcal{A}_q$ have the same Poincaré series. This means that they have the same dimension (sixteen) as vector spaces, and that the set of ordered monomials in the generators

$$\mathcal{B}_q := \{\mathbf{1}_q, \tilde{A}^\uparrow, \tilde{A}^\downarrow, \tilde{A}_\uparrow^+, \tilde{A}_\downarrow^+, \tilde{A}^\uparrow \tilde{A}^\downarrow, \tilde{A}_\downarrow^+ \tilde{A}_\uparrow^+, \dots\} \quad (1.5)$$

is a basis of \mathcal{A}_q and becomes a basis \mathcal{B} of \mathcal{A} in the limit $q \rightarrow 1$.

What is a q -deformed Clifford algebra like (1.4) good for? Can it be used to describe the same physics as its undeformed counterpart (1.1), e.g. a second-quantized fermionic system, or a different (although similar) one? What is the role of the q -deformed symmetries? Answering these questions will require a comparison not only of the algebraic structures of $\mathcal{A}, \mathcal{A}_q$, but also of their representations.

Note that the Poincaré series requirement has already an immediate consequence in representation theory: it amounts to say that the (left or right) regular representation of \mathcal{A} [i.e. the one where the carrier space is the vector space U associated to the algebra \mathcal{A} , and the elements of the algebra act on U by (left or right) multiplication] and that of \mathcal{A}_q have the same dimension, since the basis \mathcal{B} of U and the basis \mathcal{B}_q of U_q have the same number of elements.

In order to compare the representation theories it is helpful to ask first a related question: **Can we realize the generators $\tilde{A}^i, \tilde{A}_i^+$ as functions (polynomials) in a^i, a_j^+ ? And conversely?**

The answer is yes. For instance, an explicit realization of $\tilde{A}^i, \tilde{A}_i^+$ fulfilling (1.4) is through the polynomials

$$\begin{aligned} A_\downarrow^+ &:= a_\downarrow^+ & A_\uparrow^+ &:= [1 + (q^{-1} - 1)n^\downarrow]a_\uparrow^+ = q^{-n^\downarrow}a_\uparrow^+ \\ A^\downarrow &:= a^\downarrow & A^\uparrow &:= a^\uparrow[1 + (q^{-1} - 1)n^\downarrow] = a^\uparrow q^{-n^\downarrow}, \end{aligned} \quad (1.6)$$

where $n^i := a^i a_i^+$ (with *no* sum over i), $i = \uparrow, \downarrow$. (The last equalities on the right are based on the identity $(n^i)^2 = n^i$). For $q \neq 0$ the transformation (1.6) is clearly invertible; the inverse transformation allows an explicit realization of a^i, a_i^+ fulfilling (1.1) as polynomials $\tilde{a}^i, \tilde{a}_i^+$ in $\tilde{A}^i, \tilde{A}_i^+$:

$$\begin{aligned} \tilde{a}_\downarrow^+ &:= \tilde{A}_\downarrow^+ & \tilde{a}_\uparrow^+ &:= [1 + (q - 1)N^\downarrow]\tilde{A}_\uparrow^+ = q^{N^\downarrow}\tilde{A}_\uparrow^+ \\ \tilde{a}^\downarrow &:= \tilde{A}^\downarrow & \tilde{a}^\uparrow &:= \tilde{A}^\uparrow[1 + (q - 1)N^\downarrow] = \tilde{A}^\uparrow q^{N^\downarrow}, \end{aligned} \quad (1.7)$$

where $N^i := a^i a_i^+$ (with *no* sum over i), $i = \uparrow, \downarrow$. (The last equalities on the right are based on the identity $(N^\downarrow)^2 = N^\downarrow$). In a more abstract language, through the above one can define an algebra isomorphism $f : \mathcal{A}_q \rightarrow \mathcal{A}[[h]]$ ($\mathcal{A}[[h]] \equiv$ the algebra of formal power series in $h = q - 1$ with coefficients in \mathcal{A}), through

$$\begin{aligned} f(\tilde{A}^i) &= A^i & f(\tilde{A}_i^+) &= A_i^+ \\ f(\alpha\beta) &= f(\alpha)f(\beta) & \forall \alpha, \beta \in \mathcal{A} \end{aligned} \quad (1.8)$$

Now we illustrate the usefulness of deforming maps to compare the representation theories of $\mathcal{A}, \mathcal{A}_q$. We ask: can f be seen also as an *operator*

map intertwining the representations of \mathcal{A} and of \mathcal{A}_q ? In other words, given a representation of \mathcal{A} (resp. \mathcal{A}_q) on a vector space V (resp. V_q), does V (resp. V_q) carry also a representation of \mathcal{A}_q (resp. \mathcal{A})? The answer is clearly yes, since A^i, A_j^+ (resp. $\tilde{a}^i, \tilde{a}_i^+$), being polynomials in a^i, a_i^+ (resp. in $\tilde{A}^i, \tilde{A}_i^+$), result well-defined operators on V (resp. V_q)¹. Thus, the classification of the representations of \mathcal{A} (resp. \mathcal{A}_q) will determine also the classification of the representations of \mathcal{A}_q (resp. \mathcal{A}). From deforming maps one can extract also more specific informations. For instance, if we endow \mathcal{A} also with the star structure (1.2), the one that is compatible with (1.1) and the action of the compact section $U_{su}(2)$ of $U_{sl}(2)$, the corresponding star structure of \mathcal{A}_q compatible with (1.4) and the action of $U_{qsu}(2)$ exists only for real q and reads

$$(\tilde{A}^i)^{\star q} = \tilde{A}_i^+ \quad (1.9)$$

It is easy to see that (1.6) [resp. (1.7)] allows also a realization of \star_q (resp. \star) as \star (resp. \star_q). Since there is (up to unitary equivalences) a unique \star -representation of the \star -algebra \mathcal{A} and a unique \star -representation of the \star -algebra \mathcal{A}_q as one can check by direct inspection, we conclude that they correspond to each other in the above identification.

How have we found (1.6)? Does it keep track of the $U_{qsu}(2)$ symmetry?

In this report we shall present a systematic approach, based partly on the works [6, 5, 4] (see also Ref.'s [7]) to answer the latter questions for arbitrary ' q -deformed Clifford algebras'. Incidentally, the approach works not only for Clifford, but also for q -deformed Weyl algebras.

2 General framework

The general setting is the following. The undeformed Clifford algebra \mathcal{A} is covariant under some Lie algebra \mathfrak{g} and the deformed one \mathcal{A}_q under the quantum group [2] (or under a triangular deformation) $U_q\mathfrak{g}$. The undeformed algebra \mathcal{A} is generated by $\mathbf{1}, a^i, a_j^+$ fulfilling

$$\begin{aligned} a^i a^j + a^j a^i &= 0 \\ a_i^+ a_j^+ + a_j^+ a_i^+ &= 0 \end{aligned} \quad (2.1)$$

¹It is interesting to note that this is *not* true instead for q -deformations of Weyl algebras (the algebras obtained by replacing the anticommutators in relations (1.1) by commutators. The latter are in fact infinite-dimensional as vector spaces, and the corresponding realizations A^i, A_i^+ are formal power series (instead of polynomials) in a^i, a_i^+ , so strictly speaking do not belong to \mathcal{A} but just to a suitable completion of \mathcal{A} . Correspondingly, it is not guaranteed that A^i, A_i^+ can be defined as operators on the the corresponding vector spaces. In fact, it was e.g. explicitly shown [14] that there are many (inequivalent) irreducible \star -representations of the simplest $U_{qsu}(N)$ -covariant deformed Weyl \star -algebra \mathcal{A}_q , whereas just one of the corresponding undeformed partner \mathcal{A} . In Ref. [6] we re-read this result by showing that the corresponding objects $\tilde{a}^i, \tilde{a}_i^+ \in \mathcal{A}_q$ become ill-defined operators on all but one \star -representation of \mathcal{A}_q . The same might in principle occur for Clifford algebras with an infinite number of generators

$$a^i a_j^+ + a_j^+ a^i = \delta_j^i \mathbf{1}$$

and transforms under the action \triangleright of \mathbf{g} according to some law

$$x \triangleright a_i^+ = \rho(x)_i^j a_j^+ \quad x \triangleright a^i = \rho(Sx)_j^i a^j; \quad (2.2)$$

here $x \in \mathbf{g}$, $Sx = -x$ and ρ denotes some matrix representation of \mathbf{g} . Clearly a^i transform under the contragradient representation of the a_i^+ one. The action \triangleright is extended to all of $U\mathbf{g} \times \mathcal{A}$ imposing linearity, the Leibniz rule (1.3) and the law

$$(xx') \triangleright \alpha = x \triangleright (x' \triangleright \alpha). \quad (2.3)$$

As a consequence, for $x \in U\mathbf{g}$

$$x \triangleright (\alpha\alpha') \equiv \sum_i (x_{(1)}^i \triangleright \alpha)(x_{(2)}^i \triangleright \alpha'), \quad (2.4)$$

where the coproduct of $U\mathbf{g}$ $\Delta(x) = \sum_i x_{(1)}^i \otimes x_{(2)}^i$ is defined by $\Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x$ for $x \in \mathbf{g}$ and is extended to all of $U\mathbf{g}$ as an algebra homomorphism. All this is possible because the action of \mathbf{g} is manifestly compatible with the anticommutation relations (2.1). Formulae (2.2), where now we have extended S to the whole $U\mathbf{g}$ as the antipode, give also the standard extension of \triangleright to $x \in U\mathbf{g}$.

By definition the corresponding q -deformed algebra \mathcal{A}_q is generated by $\mathbf{1}_q, \tilde{A}_i^+, \tilde{A}^i$ (the generators are enumerated by the same index) fulfilling deformed anticommutation relations with a quadratic structure as in (2.1), of the form

$$\begin{aligned} P_q^{+hk} \tilde{A}_h^+ \tilde{A}_k^+ &= 0 \\ P_q^{+ij} \tilde{A}_h^k \tilde{A}^h &= 0 \\ \tilde{A}^i \tilde{A}_j^+ + P_{qjk}^{ih} \tilde{A}_h^+ \tilde{A}^k &= \delta_j^i \mathbf{1}_q. \end{aligned} \quad (2.5)$$

P_q^+, P_q are matrices with entries in \mathbb{C} . P_q is a $U_q\mathbf{g}$ -covariant deformation of the ordinary permutator P , defined by $P_{hk}^{ij} = \delta_k^i \delta_h^j$; P_q^+ is the $U_q\mathbf{g}$ -covariant deformation of the ordinary symmetric projector $P^+ = \frac{1+P}{2}$, and is a projector itself, $(P_q^+)^2 = P_q^+$. Thus, in the limit $q \rightarrow 1$, the q -deformed anticommutation relations (2.5) reduce to (2.1). Moreover, we additionally require that $\mathcal{A}, \mathcal{A}_q$ have the same Poincaré series. $U_q\mathbf{g}$ -covariance means that (2.5) are compatible with the action $\tilde{\triangleright}_q$ of $U_q\mathbf{g}$. The latter is defined on the generators by the law

$$x \tilde{\triangleright}_q \tilde{A}_i^+ = \rho_{qi}^j(x) \tilde{A}_j^+ \quad x \tilde{\triangleright}_q \tilde{A}^i = \rho_{qj}^i(S_q x) \tilde{A}^j, \quad (2.6)$$

(here $x \in U_q\mathbf{g}$, S_q is the antipode of $U_q\mathbf{g}$, ρ_q the quantum group deformation of ρ), and is extended on all of \mathcal{A}_q through a modified Leibniz rule, which we shall give below in (3.7). It is exactly the requirement of $U_q\mathbf{g}$ -covariance that determines the form of P_q^+, P_q in (2.5).

Except sometimes the case that q is a root of unity, any representation ρ of $U\mathfrak{g}$ admits a q -deformation into a representation ρ_q of $U_q\mathfrak{g}$ (in particular this implies that ρ, ρ_q have the same dimension), and the corresponding projectors P, P^+ admit $U_q\mathfrak{g}$ -covariant deformations P_q^+, P_q . However, it is not guaranteed that the corresponding relations (2.5) yield $\mathcal{A}, \mathcal{A}_q$ with the same Poincaré series.

This is guaranteed for *arbitrary* ρ only in a less general context, namely if $U_q\mathfrak{g}$ is a *triangular* deformation [1] of the Hopf algebra $U\mathfrak{g}$ (e.g. a Jordanian [13], or a Reshetikin [16] deformation); then $P_q = \hat{R} := PR$, $P_q^+ = \frac{1+\hat{R}}{2}$, where $R = (\rho_q \otimes \rho_q)\mathcal{R}$ and \mathcal{R} is a universal object belonging to $U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ called the universal R -matrix [1, 3]. Triangularity means that $\hat{R}^2 = 1$. A special case with a broad spectrum of potential physical applications is when ρ is a direct sum $\rho = \bigoplus_{\alpha=1}^M \rho'$ of M copies of a simpler representation ρ' ; if the latter describes the symmetry of some dynamical system \mathcal{S}' , ρ should describe the symmetry of the composite dynamical system \mathcal{S} obtained taking M copies of \mathcal{S}' . The M copies could correspond e.g. to different sites in some d -dimensional lattice, or (if $M = \infty$) to different space(time) points, resp. in condensed matter physics or quantum field theory.

If $U_q\mathfrak{g}$ is a quantum group in the strict sense, i.e. a *quasitriangular* but not triangular deformation of $U\mathfrak{g}$ [2], the Poincaré series condition is fulfilled essentially only if

1. $\mathfrak{g} = sl(N)[14]$, $sp(N = 2n)$ [4] and $\rho = \rho_N \equiv N$ -dimensional defining representation of \mathfrak{g} (e.g. the \mathbf{N} of $sl(N)$);
2. $\mathfrak{g} = sl(N)$ and $\rho = \bigoplus_{\alpha=1}^M \rho_N$ [4] for some integer $M > 1$.

In case 1. $P_q = q^{-1}\hat{R}_N \equiv q^{-1}PR_N$ and the projector P_q^+ is given by

$$P_q^+ = \begin{cases} \frac{1+q\hat{R}_N}{q+q^{-1}} & \text{if } \mathfrak{g} = sl(N) \\ \frac{\hat{R}_N^2 + (q^{-1-N} + q^{-1})\hat{R}_N + q^{-2-N}1}{(q+q^{-1})(q-q^{-1-N})} & \text{if } \mathfrak{g} = sp(N), \end{cases} \quad (2.7)$$

where $R_N = (\rho_{N,q} \otimes \rho_{N,q})\mathcal{R}$ and $\mathcal{R} \in U_q\mathfrak{g} \otimes U_q\mathfrak{g}$ is the so-called universal R -matrix [2] of $U_q\mathfrak{g}$. R_N is denoted as the R -matrix of $U_q\mathfrak{g}$ in the q -deformed defining representation $\rho_{N,q}$.

In case 2., contrary to the triangular case, it turns out that the resulting commutation relations between the *different* copies automatically order the M copies in a definite way, a phenomenon which we have called a ‘braided chain’ [4]; consequently the only physical lattice in which it would be reasonable to arrange the copies would be 1-dimensional. If we use greek indices $\alpha, \beta, \dots = 1, \dots, M$ to enumerate the copies in the prescribed order, then up to some free normalization factors (which we omit for the sake of simplicity) the deformed anticommutation relations (2.5) take the form,

$$\tilde{A}_{\alpha i}^+ \tilde{A}_{\beta j}^+ + q \hat{R}_{N ij}^{hk} \tilde{A}_{\beta h}^+ \tilde{A}_{\alpha k}^+ = 0$$

$$\tilde{A}^{\alpha j} \tilde{A}^{\beta i} + q \hat{R}_N^{ij} \tilde{A}^{\beta k} \tilde{A}^{\alpha h} = 0 \quad (2.8)$$

and either

$$\tilde{A}^{\alpha i} \tilde{A}_{\beta j}^+ + q^{-1} \hat{R}_N^{ih} \tilde{A}_{\beta h}^+ \tilde{A}^{\alpha k} = \delta_j^i \delta_\beta^\alpha \mathbf{1}_q \quad (2.9)$$

or

$$\tilde{A}^{\alpha i} \tilde{A}_{\beta j}^+ + \hat{R}_M^{-1\alpha\gamma} \hat{R}_N^{ih} \tilde{A}_{\gamma h}^+ \tilde{A}^{\delta k} = \delta_j^i \delta_\beta^\alpha \mathbf{1}_q, \quad (2.10)$$

with $\alpha \leq \beta$ and \hat{R}_M the braid matrix of $sl(M)$. The latter \mathcal{A}_q in fact is covariant not only under $U_q sl(N)$, but also under $U_q(sl(M) \times sl(N))$.

The explicit form of the braid matrix \hat{R}_N of $sl(N)$ is

$$\hat{R}_N = q \sum_{i=1}^N e_i^i \otimes e_i^i + \sum_{\substack{i,j=1 \\ i \neq j}}^N e_j^i \otimes e_i^j + (q - q^{-1}) \sum_{\substack{i,j=1 \\ i < j}}^N e_i^i \otimes e_j^j \quad (2.11)$$

where $q \in \mathbb{C} - 0$ and e_j^i is the matrix with all vanishing elements except a 1 at the i -th row and j -th column.

Our problem can be now formulated more technically as follows: How to determine all possible deforming maps, i.e. algebra isomorphisms (over $\mathbb{C}[[h]]$) $f : \mathcal{A}_q \rightarrow \mathcal{A}[[h]]^2$, ($h=q-1$) for the class of deformed Clifford algebras defined above?

3 Construction procedure

First note that if $\alpha \in \mathcal{A}[[h]]$ is any element of the form $\alpha = \mathbf{1} + O(h)$ and f is a deforming map, one can obtain a new one f_α by the inner automorphism

$$f_\alpha(\cdot) := \alpha f(\cdot) \alpha^{-1}; \quad (3.1)$$

actually the vanishing of the first Hochschild cohomology group [9] of \mathcal{A} implies that *all* deforming maps can be obtained from one in this manner. Therefore our problem is reduced to finding a particular one, what we are going to describe below.

Second, note that given any deforming map f and using $\tilde{\triangleright}_q$ we can draw the solid lines in the diagram

$$\begin{array}{ccc} U_q \mathbf{g} \times \mathcal{A}_q & \xrightarrow{\tilde{\triangleright}_q} & \mathcal{A}_q \\ \downarrow \text{id} \times f & & \downarrow f \\ U_q \mathbf{g} \times \mathcal{A}[[h]] & \xrightarrow{\triangleright_q} & \mathcal{A}[[h]]; \end{array} \quad (3.2)$$

we define \triangleright_q as the map making the diagram commutative (in other words $\triangleright_q := f \circ \tilde{\triangleright}_q \circ (\text{id} \otimes f^{-1})$, which will realize $\tilde{\triangleright}_q$ on $\mathcal{A}[[h]]$). One can easily realize that $\triangleright_q \neq \triangleright$, since there is no Hopf algebra isomorphism $U_q \mathbf{g} \rightarrow$

²We recall that for any algebra B $B[[h]]$ denotes the ring of formal power series in h with coefficients in B

$U\mathbf{g}[[h]]$ [3]. For each f_α in (3.1) one finds correspondingly also a different \triangleright_q , in other words by varying α one obtains all pairs (f, \triangleright_q) .

Our construction strategy will proceed in the opposite direction: we shall first determine *one* particular \triangleright_q , then the corresponding deforming map(s) f . Actually to define the latter it suffices to find generators $A^i, A_j^+ \in \mathcal{A}[[h]]$ fulfilling (2.5) and the analog of (2.6), and apply formula (1.8). We shall first show an Ansatz for A^i, A_j^+ which allows to fulfil at once the transformation law (2.6); the Ansatz is based on the properties of the ‘‘Drinfel’d twist’’ [3]. Then we shall determine in the simplest cases the free parameters appearing in the Ansatz in such a way that the commutation relations (2.5) become fulfilled.

Let us summarize the elements of our construction procedure and of the notation we shall adopt:

1. \mathbf{g} , a semisimple Lie algebra if the deformation $U_q\mathbf{g}$ we are interested in is triangular, or $sl(N)$, $sp(N)$ if the deformation $U_q\mathbf{g}$ we are interested in is a quantum group. As known, one can associate to $U\mathbf{g}$ a cocommutative Hopf algebra $H \equiv (U\mathbf{g}, \cdot, \Delta, \varepsilon, S)$; $\cdot, \Delta, \varepsilon, S$ denote the product, coproduct, counit, antipode. We shall use the Sweedler’s notation $\Delta(x) \equiv x_{(1)} \otimes x_{(2)}$: the rhs stands for a sum $\sum_i x_{(1)}^i \otimes x_{(2)}^i$ of different terms, but the symbol \sum_i is dropped. We shall denote by $H_q \equiv (U_q\mathbf{g}, \bullet, \Delta_q, \varepsilon_q, S_q, \mathcal{R})$ the deformation of H we are interested in, respectively a triangular Hopf algebra [1] or a quantum group [2]. $\bullet, \Delta_q, \varepsilon_q, S_q$ denote the deformed product, coproduct, counit, antipode, \mathcal{R} the universal R -matrix. We shall use the Sweedler’s notation (with barred indices) $\Delta_q(x) \equiv x_{(\bar{1})} \otimes x_{(\bar{2})}$.
2. An algebra isomorphism $\varphi_q : U_q\mathbf{g} \rightarrow U\mathbf{g}[[h]]$ over $\mathbf{C}[[h]]$ whose existence is proved respectively in Ref. [1, 3]: $\varphi_q(x \bullet y) = \varphi_q(x) \cdot \varphi_q(y)$.
3. A corresponding Drinfel’d twist [1, 3], i.e. an element $\mathcal{F} \equiv \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)} = \mathbf{1}^{\otimes 2} + O(h)$ of $U\mathbf{g}[[h]] \otimes U\mathbf{g}[[h]]$ such that

$$(\varepsilon \otimes \text{id})\mathcal{F} = \mathbf{1} = (\text{id} \otimes \varepsilon)\mathcal{F}, \quad \Delta_q(a) = (\varphi_q^{-1} \otimes \varphi_q^{-1})\{\mathcal{F}\Delta[\varphi_q(a)]\mathcal{F}^{-1}\}; \quad (3.3)$$

the last formula means that, up to the isomorphism φ_q , Δ_q is related to Δ by a similarity transformation.

4. $\gamma' := \mathcal{F}^{(2)} \cdot S\mathcal{F}^{(1)}$ and $\gamma := S\mathcal{F}^{-1(1)} \cdot \mathcal{F}^{-1(2)}$. Up to the isomorphism φ_q , S_q and its inverse are related to S by similarity transformations involving resp. γ and γ' .
5. The particular representation ρ_q of $U_q\mathbf{g}$ fulfilling the criteria listed after (2.6), and its classical limit (2.2).
6. The generalized Jordan-Schwinger algebra homomorphism $\sigma : U\mathbf{g}[[h]] \rightarrow \mathcal{A}[[h]]$, defined on the generators by

$$\sigma(\mathbf{1}_{U\mathbf{g}}) = \mathbf{1} \quad \sigma(x) := \rho(x)_j^i a_i^+ a^j \quad (3.4)$$

$x \in \mathfrak{g}$, and extended to the whole $U\mathfrak{g}[[h]]$ as an algebra homomorphism, $\sigma(yz) = \sigma(y)\sigma(z)$ and $\sigma(y+z) = \sigma(y) + \sigma(z)$. It is immediate to verify that this extension is possible because $\sigma([x, y]) = [\sigma(x), \sigma(y)]$. In the $su(2)$ σ takes the well-known form

$$\sigma(j_+) = a_{\uparrow}^+ a_{\downarrow}^{\downarrow}, \quad \sigma(j_-) = a_{\downarrow}^+ a_{\uparrow}^{\uparrow}, \quad \sigma(j_0) = \frac{1}{2}(a_{\uparrow}^+ a_{\uparrow}^{\uparrow} - a_{\downarrow}^+ a_{\downarrow}^{\downarrow}). \quad (3.5)$$

7. The deformed Jordan-Schwinger algebra homomorphism $\sigma_q : U_q\mathfrak{g} \rightarrow \mathcal{A}[[h]]$, defined by $\sigma_q := \sigma \circ \varphi_q$.
8. The $*$ -structures $*, *_q, \star, \star_q$ in $H, H_q, \mathcal{A}, \mathcal{A}_q$, if $\mathcal{A}, \mathcal{A}_q$ are $*$ -algebras transforming respectively under the Hopf $*$ -algebras H, H_q with the compatibility condition

$$(x \triangleright_q a)^{\star_q} = S_q^{-1}(x^{\star_q}) \triangleright_q a^{\star_q}. \quad (3.6)$$

As anticipated, our first step is to guess a realization \triangleright_q of $\tilde{\triangleright}_q$ on $\mathcal{A}[[h]]$, instead of \mathcal{A}_q . This requires fulfilling

$$(xy) \triangleright_q a = x \triangleright_q (y \triangleright_q a) \quad x \triangleright_q (ab) = (x_{(1)} \triangleright_q a)(x_{(2)} \triangleright_q b) \quad (3.7)$$

for any $x, y \in U_q\mathfrak{g}$, $a, b \in \mathcal{A}_q$; these are the conditions i.e. characterizing a module algebra [also in the undeformed case, see formulae (1.3)] There is a simple way to find such a realization, namely by setting

$$x \triangleright_q a := \sigma_q(x_{(1)}) a \sigma_q(S_q x_{(2)}); \quad (3.8)$$

it is easy to check that (3.7) are indeed fulfilled using the basic axioms characterizing the coproduct, counit, antipode in a generic Hopf algebra. The guess has been suggested by the undeformed case, where the same conditions and realization are obtained for $U\mathfrak{g}, \mathcal{A}, \triangleright$ if in the two previous formulae we just erase the suffix $_q$ and replace $\Delta_q(x) \equiv x_{(1)} \otimes x_{(2)}$ with the cocommutative coproduct $\Delta(x) \equiv x_{(1)} \otimes x_{(2)}$.

Our second step is to realize elements $A_i^{\pm}, A_j^{\pm} \in \mathcal{A}[[h]]$ that transform under the action \triangleright_q defined by (3.8) as $\tilde{A}^i, \tilde{A}_j^{\pm}$ do under $\tilde{\triangleright}_q$ [see (2.6), namely

$$x \triangleright_q A_i^+ = \rho_{qi}^j(x) A_j^+ \quad x \triangleright_q A^i = \rho_{qj}^i(S_q x) A^j, \quad (3.9)$$

Note that a^i, a_j^{\pm} do *not* transform in this way. In Ref. [6] we proved that the following objects do:

$$\begin{aligned} A_i^+ &:= u \sigma(\mathcal{F}^{(1)}) a_i^+ \sigma(S \mathcal{F}^{(2)} \gamma) u^{-1} \\ A^i &:= v \sigma(\gamma' S \mathcal{F}^{-1(2)}) a^i \sigma(\mathcal{F}^{-1(1)}) v^{-1}; \end{aligned} \quad (3.10)$$

the result holds for any choice of g -invariant elements $u, v = \mathbf{1} + O(h)$ in $\mathcal{A}[[h]]$, in particular for $u = v = \mathbf{1}$.

The third step is to fix u, v in such a way that the deformed commutation relations (2.5) are fulfilled. One can easily show that the latter may fix at most the product uv^{-1} . In the case that $U_q \mathfrak{g}$ is triangular, we showed in Ref. [6] that they require $uv^{-1} = 1$. In the case that $U_q \mathfrak{g}$ is the quantum group $U_q sl(N)$ and $\rho = \rho_N$ we proved [5] that the deformed commutation relations require

$$uv^{-1} = \frac{\Gamma(n+1)}{\Gamma_{q^2}(n+1)}. \quad (3.11)$$

Here Γ is Euler's γ -function, Γ_{q^2} its q -deformation characterized by $\Gamma_{q^2}(x+1) = (x)_{q^2} \Gamma_{q^2}(x)$, $n := \sum_i a^i a_i^+$. We stress that the above solutions regard the case of ρ being the defining representation ρ_N of $sl(N)$. We have yet no formula yielding the right uv^{-1} , if any, necessary to fulfil the (2.5) in the other cases. However it is important to recall that [5] in general the (2.5) translate into conditions on uv^{-1} where the Drinfel'd twist \mathcal{F} appears only through the so-called 'coassociator'

$$\phi := [(\Delta \otimes \text{id})(\mathcal{F}^{-1})](\mathcal{F}^{-1} \otimes \mathbf{1})(\mathbf{1} \otimes \mathcal{F})[(\text{id} \otimes \Delta)(\mathcal{F})]. \quad (3.12)$$

ϕ is known, unlike \mathcal{F} , for which up to now there is an existence proof. This makes the above conditions explicit and allows to search uv^{-1} in the general case, if it exists. The explicit expression for ϕ is

$$\phi = \lim_{x_0, y_0 \rightarrow 0^+} \left\{ x_0^{-\hbar t_{12}} \vec{P} \exp \left[-\hbar \int_{x_0}^{1-y_0} dx \left(\frac{t_{12}}{x} + \frac{t_{23}}{x-1} \right) \right] y_0^{\hbar t_{23}} \right\} \quad (3.13)$$

where $t = \Delta(\mathcal{C}) - \mathbf{1} \otimes \mathcal{C} - \mathcal{C} \otimes \mathbf{1}$, \mathcal{C} denoting the quadratic Casimir of $U \mathfrak{g}$, $t_{12} = t \otimes \mathbf{1}$, $t_{23} = \mathbf{1} \otimes t$, and the symbol \vec{P} means that we must understand a path-ordered integral in the variable x . Note that $\phi = \mathbf{1}^{\otimes 3} + O(\hbar^2)$. (In the triangular case, on the contrary, $\phi \equiv \mathbf{1}^{\otimes 3}$.)

Finally, the residual freedom in the choice of u, v is partially fixed if $H, H_q, \mathcal{A}, \mathcal{A}_q$ are matched (Hopf) \ast -algebras and we make the additional requirement that \star realizes in $\mathcal{A}[[\hbar]]$ the \star_q of \mathcal{A}_q . For instance, if $(a^i)^\star = a_i^+$ and $q \in \mathbb{R}^+$ this means

$$(A^i)^\star = A_i^+, \quad (3.14)$$

and is fulfilled if we take $u = v^{-1}$.

In general, one can show that the knowledge of $(\rho \otimes \text{id})\mathcal{F}$ is sufficient to determine the A^i, A_i^+ of formulae (3.10) completely. In the $\mathfrak{g} = sl(2)$ case, with ρ being the fundamental representation, $(\rho \otimes \text{id})\mathcal{F}$ is explicitly known [17]. Taking $u = v^{-1}$ one finally finds [6] the result (1.6) we had anticipated.

Above we have determined in $\mathcal{A}[[\hbar]]$ one particular realization A^i, A_j^+ and \triangleright_q of the generators $\tilde{A}^i, \tilde{A}_j^+$ and of the quantum group action. Its main feature is that the \mathfrak{g} -invariant ground state $|0\rangle$ as well as the first excited states $a_i^+|0\rangle$ of the classical Fock space representation are also respectively $U_q \mathfrak{g}$ -invariant ground state $|0_q\rangle$ and first excited states $A_i^+|0_q\rangle$ of the deformed Fock space representation.

According to eq. (3.1) all the other realizations are of the form

$$A^{\alpha i} = \alpha A^i \alpha^{-1} \quad A_{\alpha i}^+ = \alpha A_i^+ \alpha^{-1}, \quad (3.15)$$

with $\alpha = \mathbf{1} + O(h) \in \mathcal{A}[[h]]$. They are manifestly covariant under the realization $\triangleright_{h,\alpha}$ of the $U_q \mathbf{g}$ -action defined by

$$x \triangleright_{h,\alpha} a := \alpha \sigma_q(x_{(\bar{1})}) a \sigma_q(x_{(\bar{2})}) \alpha^{-1}. \quad (3.16)$$

For these realizations the deformed ground state in the Fock space representation reads $|0_q\rangle = \alpha|0\rangle$; if $\alpha|0\rangle \neq |0\rangle$ the \mathbf{g} -invariant ground state and first excited states of the classical Fock space representation do not coincide with their deformed counterparts.

4 Ordinary vs. q -deformed invariants

We have introduced two actions on $\mathcal{A}[[h]]$:

$$\triangleright : U \mathbf{g} \times \mathcal{A}[[h]] \rightarrow \mathcal{A}[[h]], \quad \triangleright_q : U_q \mathbf{g} \times \mathcal{A}[[h]] \rightarrow \mathcal{A}[[h]]. \quad (4.1)$$

Their respective invariant subalgebras $\mathcal{A}^{inv}[[h]]$, $\mathcal{A}_q^{inv}[[h]]$ are defined by

$$\mathcal{A}_q^{inv}[[h]] := \{I \in \mathcal{A}[[h]] \mid x \triangleright_q I = \varepsilon_q(x) I \quad \forall x \in U_q \mathbf{g}\} \quad (4.2)$$

and by the analogous equation where all suffices $_q$ are erased. What is the relation between them? It is easy to prove that [5]

$$\mathcal{A}_q^{inv}[[h]] = \mathcal{A}^{inv}[[h]]. \quad (4.3)$$

In other words invariants under the \mathbf{g} -action \triangleright are also $U_q \mathbf{g}$ -invariants under \triangleright_q , and conversely, although in general \mathbf{g} -covariant objects (tensors) and $U_q \mathbf{g}$ -covariant ones do not coincide in general!

Let us introduce in the vector space $\mathcal{A}^{inv}[[h]] = \mathcal{A}_q^{inv}[[h]]$ bases I^1, I^2, \dots and I_q^1, I_q^2, \dots . It is immediate to realize that we can choose the I^n as homogeneous, normal-ordered polynomials in a^i, a_j^+ and I_q^n as homogeneous, normal-ordered polynomials in A^i, A_j^+ , since \triangleright acts linearly without changing the degrees in a^i and a_j^+ , and \triangleright_q acts linearly without changing the degrees in A^i and A_j^+ . Explicitly,

$$\begin{aligned} I^1 &= a_i^+ a^i & I_q^1 &= A_i^+ A^i \\ I^2 &= d^{ijk} a_i^+ a_j^+ a_k^+ & I_q^2 &= D^{ijk} A_i^+ A_j^+ A_k^+ \\ I^3 &= d'_{kji} a^i a^j a^k & I_q^3 &= D'_{kji} A^i A^j A^k \\ I^4 &= \dots & I_q^4 &= \dots \end{aligned} \quad (4.4)$$

where the numerical coefficients d, d', \dots form \mathbf{g} -isotropic tensors and the numerical coefficients D, D' the corresponding $U_q \mathbf{g}$ -isotropic tensors. In

the quantum group cases considered in Section 2 it is possible to show that $I_q^1 \neq I^1$:

$$I_q^1 = \frac{q^{-2I^1} - 1}{q^{-2} - 1} \quad (4.5)$$

In general $I_q^n \neq I^n$, although $I_q^n = I^n + O(h)$. The proposition (4.3) implies in particular

$$I_q^n = g^n(\{I^m\}, h) = k^n(\{a^i, a_j^+\}, h). \quad (4.6)$$

What do the functions g^n, k^n look like?

In Ref. [5] we have found universal formulae yielding the k^n 's. The latter turn out to be polynomials in a^i, a_i^+ of degree higher than the degree in A^i, A_i^+ [this can be easily worked e.g. for the invariant I_q given in (4.5)], and the degree difference grows very fast with the number of these generators. It is remarkable that in these universal formulae the twist \mathcal{F} appears only through the coassociator ϕ ; therefore all the k^n can be worked out explicitly.

In the case that the Hopf algebra H_q is not a genuine quantum group, but triangular, the coassociator as well as u, v are trivial and one finds $I_q^n = I^n$.

5 Final remarks, outlook and conclusions

We have shown how one can realize a deformed $U_q\mathfrak{g}$ -covariant Clifford algebra \mathcal{A}_q within the undeformed one $\mathcal{A}[[h]]$. Given a representation (π, V) of \mathcal{A} on a vector space V , does it provide also a representation of \mathcal{A}_q ? In other words, can one interpret the elements of \mathcal{A}_q as operators acting on V , if the elements of \mathcal{A} are? If so, which specific role play the elements A^i, A_i^+ of $\mathcal{A}[[h]]$?

Repeating the arguments presented in the introduction for the toy-model, one can conclude that the answer to the first question is always positive, at least for finite-dimensional Clifford algebras. In particular, when q is real and the real structure (1.9) is chosen this allows to represent the q -deformed Clifford algebra \mathcal{A}_q on the standard Fock space of the original algebra \mathcal{A} ; in a particle-physics interpretation no exotic statistics are then involved, but just the ordinary Fermi-Dirac characterizing fermions. Only, A^i, A_j^+ do not annihilate/create the undeformed states.

On the other hand quadratic commutation relations of the type (2.5) mean that A_i^+, A^i can be interpreted as creators and annihilators of some excitations; a glance at (3.10), (3.15) shows that these are not the undeformed excitations, but some ‘collective’ ones³. The last point is: what could the latter be good for. As an Hamiltonian H of the system we can choose a simple combination of the $U_q\mathfrak{g}$ -invariants I_q^n of section 4;

³The idea that deformed excitations should consist of a compound of ordinary ones is not new, both for fermions and for bosons: see for instance Ref.'s [12]

the Hamiltonian is $U_q\mathfrak{g}$ -invariant and has a simple polynomial structure in the composite operators A^i, A_j^+ . H is also \mathfrak{g} -invariant, but has a higher degree polynomial structure (or more generally a non-polynomial structure if \mathcal{A} has an infinite number of generators) in the undeformed generators a^i, a_j^+ . This suggests that the use of the A^i, A_j^+ instead of the a^i, a_j^+ should simplify the resolution of the corresponding dynamics.

The results presented in the previous paragraphs could in principle be applied to models in quantum field theory or condensed matter physics by choosing representations ρ which are the direct sum of many copies of the same fundamental representation ρ_d ; this is what we have addressed in Ref. [4]. The different copies could correspond respectively to different space(time)-points or crystal sites.

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